Ex 1.3

Q1 Factorize the following

(i) $a^2 + 4b^2$

$= a^2 - (4b^2(-1))$

$= a^2 - 4b^2i^2$

$= a^2 - (2bi)^2$

$= (a + 2bi)(a - 2bi)$

(ii) $9a^2 + 16b^2$

$= (3a)^2 - (4bi)^2$

$= (3a - 4bi)(3a + 4bi)$

(iii) $3x^2 + 3y^2$

$= 3(x^2 + y^2)$

$= 3(x^2 - y^2(-1))$

$= 3(x^2 - (yi)^2)$

$= 3(x - iy)(x + iy)$

(iv) $144x^2 + 225y^2$

Sol:

$= 9(16x^2 + 25y^2)$

$= 9(16x^2 - 25y^2(-1))$

$= 9((4x)^2 - (5yi)^2)$

$= 9(4x - i5y)(4x + i5y)$

(v) $z^2 - 2iz - 1$

$= z^2 - 2iz + (-1)$

$= z^2 - 2iz + i^2$

$= z^2 - iz - iz + i^2$

$= z(z-i) - i(z-i)$

$= (z-i)(z-i)$

(vi) $z^2 + 6z + 13$

$= z^2 + 6z + 13$

$a = 1, \quad b = 6, \quad c = 13$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(13)}}{2(1)}$$

$$z = \frac{-6 \pm \sqrt{36 - 52}}{2}$$

$$z = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2}$$

$$z = \frac{\cancel{2}(-3 \pm 2i)}{\cancel{2}}$$

$z = -3 \pm 2i$

$z = -3 + 2i, \quad z = -3 - 2i$

So,

$= (z + 3 - 2i)(z + 3 + 2i)$

vii $z^2 + 4z + 5$

Sol:-

$a = 1, \quad b = 4, \quad c = 5$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm \sqrt{4i^2}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= \frac{2(-2 \pm i)}{2}$$

$$= (-2 \pm i)$$

$z = -2 + i, \quad z = -2 - i$

$(z + 2 - i)(z + 2 + i)$

(viii) $2z^2 - 22z + 65$

Sol:-

$a = 2, \quad b = -22, \quad c = 65$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$z = \frac{-(-22) \pm \sqrt{(-22)^2 - 4(2)(65)}}{2(2)}$$

$$= \frac{22 \pm \sqrt{484 - 520}}{4}$$

$$= \frac{22 \pm \sqrt{-36}}{4}$$

$$= \frac{22 \pm \sqrt{36i^2}}{4}$$

$$= \frac{22 \pm 6i}{4}$$

$$= \frac{2(11 \pm 3i)}{4}$$

$$= \frac{11 \pm 3i}{2}$$

$$z = \frac{11 + 3i}{2}, \quad z = \frac{11 - 3i}{2}$$

$$\left(z - \frac{11 - 3i}{2}\right)\left(z - \frac{11 + 3i}{2}\right)$$

Q2 Factorize the following polynomials into its linear factors:

(i) $z^3 + 8$

Sol:-

As $z^3 + (2)^3$

$= (z+2)(z^2 - 2z + 4) \quad (\because a^3 + b^3 = (a+b)(a^2 - ab + b^2))$

$(z^2 - 2z + 4)$

$a = 1, \quad b = -2, \quad c = 4$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm \sqrt{4 \times 3 i^2}}{2}$$

$$= \frac{2 \pm 2\sqrt{3} i}{2}$$

$$= \frac{\cancel{2}(1 \pm \sqrt{3} i)}{\cancel{2}}$$

$(z+2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)$

(ii) z^3+27

Sol:- $z^3+(3)^3$

$= (z+3)(z^2-3x+9)$

Now

z^2-3x+9

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2-4(1)(9)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9-36}}{2}$$

$$= \frac{3 \pm \sqrt{-27}}{2}$$

$$= \frac{3 \pm \sqrt{9 \times 3 i^2}}{2}$$

$$= \frac{3 \pm 3\sqrt{3}i}{2}$$

$$(z+3)\left(z-\frac{3}{2}+\frac{3\sqrt{3}i}{2}\right)\left(z-\frac{3}{2}+\frac{3\sqrt{3}i}{2}\right)$$

(iii) $z^3 - 2z^2 + 16z - 32$

Sol:-

Check for rational roots by testing possible factors of the constant terms ($\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$)

Try $z = 2$

$(2)^3 - 2(2)^2 + 16(2) - 32 = 0$

$8 - 8 + 32 - 32 = 0$

$0 = 0$

So $z = 2$ is a root, that's mean $z - 2$ is a factor.

Using synthetic division

$$\begin{array}{c|cccc} & 1 & -2 & 16 & -32 \\ 2 & \downarrow & +2 & 0 & 32 \\ \hline & 1 & 0 & 16 & \boxed{0} \end{array}$$

Quotient: $z^2 + 0z + 16 =$

$= z^2 + 16$

$= z^2 + 16 = z^2 - (4i)^2$

$= (z + 4i)(z - 4i)$

So,

$(z - 2)(z + 4i)(z - 4i)$.

(iv) $z^4 + 21z^2 - 100$

Sol:-

Let $u = z^2$

$u^2 = z^4$

$= u^2 + 21u - 100$

$= u^2 + 25u - 4u - 100$

$= u(u+25) - 4(u+25)$

$= (u+25)(u-4)$

$= (z^2+25)(z^2-4)$

$\therefore z^2 + 25 = z^2 - 25(-1)$

$= z^2 - 25i^2$

$= z^2 - (5i)^2 = (z+5i)(z-5i)$

So,

$z^4 + 21z^2 - 100 = (z+5i)(z-5i)(z^2-4)$

$= (z+5i)(z-5i)(z+2)(z-2)$

(v) $z^4 - 16$

Sol:- $(z^2)^2 - (4)^2$

$= (z^2-4)(z^2+4)$

$= (z-2)(z+2)(z^2-(2i)^2)$

$= (z-2)(z+2)(z-2i)(z+2i)$

(vi) $z^4 + 3z^2 - 4$

Sol: We are given quartic (degree 4) polynomial:

$$z^4 + 3z^2 - 4$$

Let $u = z^2$

$u^2 = z^4$

$= u^2 + 3u - 4$

$= u^2 + 4u - u - 4$

$= u(u+4) - 1(u+4)$

$= (u+4)(u-1)$

$= (z^2+4)(z^2-1)$

$= (z^2 - (2i)^2)(z+1)(z-1)$

$= (z+2i)(z-2i)(z+1)(z-1)$

(vii) $z^4 + 5z^2 + 6$

Sol:- $z^4 + 5z^2 + 6$

Let : $u = z^2$.

$u^2 = z^4$.

$= u^2 + 5u + 6$

$= u^2 + 3u + 2u + 6$

$= u(u+3) + 2(u+3)$

$= (u+3)(u+2)$

$= (z^2+3)(z^2+2)$

$=$

$= (z^2 - (\sqrt{3}i)^2)(z^2 - (\sqrt{2}i)^2)$

$= (z - \sqrt{3}i)(z + \sqrt{3}i)(z - \sqrt{2}i)(z + \sqrt{2}i)$

(viii) $z^4 - 32z^2 - 3969$

Sol:-

Let $u = z^2$

$u^2 = z^4$

$= u^2 - 32u - 3969$

$= u^2 - 81u + 49u - 3969$

$= u(u - 81) + 49(u - 81)$

$= (u - 81)(u + 49)$

$(z^2 - 81)(z^2 + 49)$

$= (z^2 - (9)^2)(z^2 - (7i)^2)$

$= (z - 9)(z + 9)(z - 7i)(z + 7i)$

Q3: Find the roots of $z^4 + 7z^2 - 144 = 0$ and hence express it as a product of linear factors.

Sol:- We are given the equation

$z^4 + 7z^2 - 144 = 0$

Let $u = z^2$

and $u^2 = z^4$

$u^2 + 7u - 144 = 0$

Using quadratic formula

$$u = \frac{-7 \pm \sqrt{7^2 - 4(1)(-144)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49+576}}{2}$$

$$= \frac{-7 \pm \sqrt{625}}{2}$$

$$= \frac{-7 \pm 25}{2}$$

$$u = \frac{-7+25}{2}, \quad u = \frac{-7-25}{2}$$

$$u = \frac{18}{2}, \quad u = \frac{-32}{2}$$

$$\boxed{u = 9}, \quad \boxed{u = -16}$$

$$z^2 = 9, \quad z^2 = -16$$

$$z = \pm 3, \quad z = \pm 4i$$

So roots will be

$$(z+3)(z-3)(z+4i)(z-4i).$$

Q4: Solve the following complex quadratic equations by completing square method.

i) $2z^2 - 3z + 4 = 0$

Sol:- $2z^2 - 3z + 4 = 0$

Dividing both sides by "2"

$$z^2 - \frac{3}{2}z + 2 = 0$$

$$\Rightarrow z^2 - \frac{3}{2}z = -2$$

Add $\left(\frac{3}{4}\right)^2$ on both sides

$$z^2 - 2(z)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 = -2 + \left(\frac{3}{4}\right)^2$$

$$\left(z - \frac{3}{4}\right)^2 = -2 + \frac{9}{16}$$

$$\left(z - \frac{3}{4}\right)^2 = \frac{-32 + 9}{16}$$

$$\left(z - \frac{3}{4}\right)^2 = \frac{-23}{16}$$

$$\sqrt{\left(z - \frac{3}{4}\right)^2} = \sqrt{\frac{-23}{16}}$$

$$z - \frac{3}{4} = \frac{\sqrt{(23)i^2}}{4}$$

$$z = \frac{\pm\sqrt{23}\,i}{4} + \frac{3}{4}$$

$$z = \frac{\pm\sqrt{23}\,i + 3}{4}$$

$$z = \frac{3 \pm i\sqrt{23}}{4}$$

(ii) $z^2 - 6z + 30 = 0$

Sol:- $z^2 - 6z + 30 = 0.$

$$z^2 - 6z = -30$$

Add 3^2 on Both sides

$$z^2 - 2(3)z + (3)^2 = -30 + (3)^2$$

$$(z-3)^2 = -30 + 9$$

$$(z-3)^2 = -21$$

$$\sqrt{(z-3)^2} = \sqrt{-21}$$

$$z - 3 = \pm\sqrt{21}\,i$$

$$z = 3 \pm \sqrt{21}\,i$$

(iii) $3z^2 - 18z + 50 = 0$

Sol:- $3z^2 - 18z + 50 = 0$

Divided both sides by "3"

$$\frac{3z^2}{3} - \frac{18}{3}z + \frac{50}{3} = 0$$

$$z^2 - 6z = -\frac{50}{3}$$

Add "3^2" on both sides

$$z^2 - 2(3)x + 3^2 = -\frac{50}{3} + (3)^2$$

$$(z^2 - 3)^2 = -\frac{50}{3} + 9$$

$$(z-3)^2 = \frac{-50+27}{3}$$

$$(z-3)^2 = \frac{-23}{3}$$

$$\sqrt{(z-3)^2} = \sqrt{-\frac{23}{3}}$$

$$(z-3) = \pm i\sqrt{\frac{23}{3}}$$

$$z = \pm i\frac{\sqrt{23}}{\sqrt{3}} + 3$$

$$z = \frac{3 \pm i\sqrt{23}}{3}$$

(iv) $z^2 + 4z + 13 = 0$

Sol:-

$$z^2 + 4z + 13 = 0$$

$$z^2 + 4z = -13$$

Add 2^2 on B.S.

$$z^2 + 2(2)z + (2)^2 = -13 + (2)^2$$

$$(z+2)^2 = -13 + 4$$

$$(z+2)^2 = -9$$

$$\sqrt{(z+2)^2} = \sqrt{-9}$$

$z+2 = \pm 3i$

$z = \pm 3i - 2$

$z = -2 \pm 3i$

(v) $2z^2 + 6z + 9 = 0$

Sol:-

$2z^2 + 6z + 9 = 0$

Dividing both sides by "2"

$\frac{2z^2}{2} + \frac{6z}{2} + \frac{9}{2} = 0$

$z^2 + 3z + \frac{9}{2} = 0$

$z^2 + 3z = -\frac{9}{2}$

$z^2 + 2(z)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 = -\frac{9}{2} + \left(\frac{3}{2}\right)^2$

$\left(z + \frac{3}{2}\right)^2 = -\frac{9}{2} + \frac{9}{4}$

$\left(z + \frac{3}{2}\right)^2 = \frac{-18+9}{4}$

$\left(z + \frac{3}{2}\right)^2 = -\frac{9}{4}$

$\sqrt{\left(z + \frac{3}{2}\right)^2} = \sqrt{-\frac{9}{4}}$

$z + \frac{3}{2} = \pm \frac{3}{2}i$

$$z = \pm\frac{3}{2}i - \frac{3}{2}$$

$$z = \frac{\pm 3i - 3}{2}$$

$$z = \frac{-3}{2} \pm \frac{3i}{2}$$

(vi) $3z^2 - 5z + 7 = 0$

Sol:-

$$3z^2 - 5z + 7 = 0$$

Dividing both sides by 3

$$\frac{3z^2}{3} - \frac{5}{3}z + \frac{7}{3} = 0$$

$$z^2 - \frac{5}{3}z = -\frac{7}{3}$$

$$(z)^2 - 2(z)\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 = -\frac{7}{3} + \left(\frac{5}{6}\right)^2$$

$$\left(z - \frac{5}{6}\right)^2 = \frac{-7}{3} + \frac{25}{36}$$

$$\left(z - \frac{5}{6}\right)^2 = \frac{-84 + 25}{36}$$

$$\sqrt{\left(z - \frac{5}{6}\right)^2} = \sqrt{\frac{-59}{36}}$$

$$z - \frac{5}{6} = \frac{\pm\sqrt{59}\,i}{6}$$

$$z = \frac{5}{6} \pm \frac{\sqrt{59}}{6} i$$

Q5 Solve the following equations.

(i) $2z^4 - 32 = 0$

Sol:-

$$2z^4 - 32 = 0$$

$$\Rightarrow 2z^4 = 32$$

$$\Rightarrow z^4 = 16$$

$$\sqrt{{z^2}^2} = \sqrt{16}$$

$$z^2 = \pm 4$$

$$z^2 = 4 \quad ; \quad z^2 = -4$$

$$\sqrt{z^2} = \sqrt{4} \quad ; \quad \sqrt{z^2} = \sqrt{-4}$$

$$z = \pm 2 \quad , \quad z = \pm 2i$$

$$(2, -2, 2i, -2i)$$

(ii) $3z^5 - 243z = 0$

Sol:- $3z(z^4 - 81) = 0$

$$3z = 0 \quad ; \quad z^4 - 81 = 0$$

$$\boxed{z = 0} \qquad z^4 = 81$$

$$\sqrt{{z^2}^2} = \sqrt{81}$$

$$z^2 = \pm 9$$

$z^2 = 9 \quad ; \quad z^2 = -9$

$\sqrt{z^2} = \sqrt{9} \quad ; \quad \sqrt{z^2} = \sqrt{-9}$

$z = \pm 3 \quad ; \quad z = \pm 3i$

$\{0, 3, -3, 3i, -3i\}$

(iii) $5z^5 - 5z = 0$

Sol:- $5z(z^4 - 1) = 0$

$5z = 0 \quad , \quad z^4 - 1 = 0$

$z = 0 \quad , \quad z^4 = 1$

$\sqrt{z^4} = \sqrt{1}$

$z^2 = \pm 1$

$z^2 = 1 \quad , \quad z^2 = -1$

$\sqrt{z^2} = \sqrt{1} \quad , \quad \sqrt{z^2} = \sqrt{-1}$

$z = \pm 1 \quad , \quad z = \pm i$

$\{0, 1, -1, i, -i\}$

(iv) $z^3 - 5z^2 + z - 5 = 0$

Sol:-

$z^3 - 5z^2 + z - 5 = 0$

$z^2(z-5) + 1(z-5) = 0$

$(z-5)(z^2+1) = 0$

$z-5=0$, $z^2+1=0$

$z=5$, $z^2=-1$

, $\sqrt{z^2}=\sqrt{-1}$

, $z=\sqrt{i^2}$

, $z=\pm i$

$\{5, +i, -i\}$

(v) $4z^4-25z^2-21=0$

Sol:-

Let $z^2=u$

and $z^4=u^2$

$4u^2-25u-21=0$

$a=4$, $b=-25$, $c=-21$

By using formula

$$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$u=\frac{-(-25)\pm\sqrt{(-25)^2-4(4)(-21)}}{2(4)}$$

$$u=\frac{25\pm\sqrt{625+336}}{8}$$

$$u=\frac{25\pm\sqrt{961}}{8}.$$

$$u = \frac{25 \pm 31}{8}$$

$$u = \frac{25+31}{8} \quad ; \quad u = \frac{25-31}{8}$$

$$u = \frac{56}{8} \quad ; \quad u = \frac{-6}{8}$$

$$u = 7 \quad ; \quad u = \frac{-3}{4}$$

$$z^2 = 7 \quad ;$$

$$\sqrt{z^2} = \sqrt{7} \quad ; \quad \sqrt{z^2} = \sqrt{\frac{-3}{4}}$$

$$z = \pm\sqrt{7} \quad ; \quad z = \pm \frac{\sqrt{3}}{2} i$$

$$\left\{ \sqrt{7}, -\sqrt{7}, \frac{\sqrt{3}}{2} i, -\frac{\sqrt{3}}{2} i \right\}$$

(vi) $z^3 + z^2 + z + 1 = 0$

Sol:- $z^2(z+1) + 1(z+1) = 0$

$(z+1)(z^2+1) = 0$

$z+1 = 0, \quad z^2+1 = 0$

$z = -1 \quad ; \quad z^2 = -1$

$\sqrt{z^2} = \sqrt{-1}$

$z = \pm i$

$\{-1, +i, -i\}$

Q6: Find a polynomial $P(z)$ of degree 3 with zeros $3, -2i, 2i$ and satisfying $P(1) = 20$

Sol:-

We are given

Degree 3 polynomial ; $P(z)$

Zeros : $3, -2i, 2i$

Condition : $P(1) = 20$

The general form will be (with constant a).

$$P(z) = a(z-3)(z+2i)(z-2i)$$

$\therefore (z+2i)(z-2i) = z^2+4$

So, $P(z)$ become

$$P(z) = a(z-3)(z^2+4)$$

using condition $P(1) = 20$

$$P(1) = a(1-3)((1)^2+4)$$

$$20 = a(-2)(1+4)$$

$$20 = a(-2)(5)$$

$$20 = -10a$$

$$\boxed{a = -2}$$

$$P(z) = -2(z-3)(z^2+4)$$
$$= -2(z^3-3z^2+4z-12)$$
$$= -2z^3+6z^2-8z+24$$

Q7. Find a polynomial $P(z)$ of degree 4 with zeros $2i, -2i, 1, -1$ and satisfying $P(2) = 240$

Sol:-

We are given

Degree 4 polynomial $= P(z)$

zeros: $2i, -2i, 1, -1$

Condition: $P(2) = 240$

Form factors with constant a

$$P(z) = a(z-2i)(z+2i)(z-1)(z+1)$$

$\therefore$

$$(z-2i)(z+2i) = z^2+4$$
$$(z-1)(z+1) = z^2-1$$

So,

$$P(z) = a(z^2+4)(z^2-1)$$

Use the condition $P(2) = 240$

$$P(2) = a((2)^2+4)((2)^2-1)$$
$$240 = a(4+4)(4-1)$$
$$240 = a(8)(3)$$

$240 = a(24)$

$\boxed{a = 10}$

$P(z) = 10(z^2+4)(z^2-1)$

$= 10(z^4+4z^2-z^2-4)$

$= 10(z^4+3z^2-4)$

$= 10z^4+30z^2-40$

Q8: Find a polynomial $P(z)$ of degree 4 with zeros $4, -4, (1+i), (1-i)$ and satisfying $P(2) = 72$.

Sol:-

We are given

Degree 4 polynomial: $P(z)$

Zeros: $4, -4, 1+i, 1-i$

Condition : $P(2) = 72$

Form factors with constant a.

$P(z) = a(z-4)(z+4)(z-(1+i))(z-(1-i))$

$\therefore (z-4)(z+4) = z^2-16$

$: (z-(1+i))(z-(1-i))$

$= ((z-1)-i)((z-1)+i)$

$= (z-1)^2-(i^2)$

$= z^2+1-2z+1$

$= z^2-2z+2$

Now $P(z)$ become

$$P(z) = a(z^2-16)(z^2-2z+2)$$

Use the condition $P(2)=72$

$$P(2) = a((2)^2-16)((2)^2-2(2)+2)$$

$$72 = a(4-16)(4-4+2)$$

$$72 = a(-12)(2)$$

$$72 = -24a$$

$$\boxed{a = -3}$$

$$P(z) = -3(z^2-16)(z^2-2z+2)$$

$$P(z) = -3(z^4-2z^3-14z^2+32z-32)$$

$$P(z) = -3z^4+6z^3+42z^2-96z+96$$