Exercise 1.4

Q1: Find the cube root of :

i) 8

Sol:- Let x be the cube root

$$x^3 = 8$$

$$x^3 - 8 = 0$$

$$x^3 - (2)^3 = 0$$

$$(x-2)(x^2+2x+4)=0 \quad \because a^3-b^3=(a-b)(a^2+ab+b^2)$$

$x-2=0$, $x^2+2x+4=0$

$x=2$, $a=1$, $b=2$, $c=4$

$$, \quad x = \frac{-2 \pm \sqrt{(2)^2-4(1)(4)}}{2(1)}$$

$$, \quad x = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$, \quad x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$, \quad x = \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2(-1 \pm \sqrt{-3})}{2}$$

$$x = 2\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$$

$$x = 2\left(\frac{-1 \pm \sqrt{3}i}{2}\right)$$

$$\{2, 2\omega, 2\omega^2\}$$

(ii) -8

Sol:- Let x be the cube root

$$x^3 = -8$$

$$x^3 + 8 = 0$$

$$x^3 + 2^3 = 0$$

$$(x+2)(x^2-2x+4) = 0 \quad \because (a+b)(a^2-ab+b^2)$$

$$x+2=0 \quad ; \quad x^2-2x+4=0$$

$$x=-2 \quad , \quad a=1, \; b=-2 \quad c=4$$

$$; \quad x = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(4)}}{2(1)}$$

$$; \quad x = \frac{2 \pm \sqrt{4-16}}{2}$$

$$x = \frac{2 \pm \sqrt{-12}}{2}$$

$$x = \frac{2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2(1 \pm \sqrt{3}i)}{2}$$

$$x = \frac{-2(-1 \pm \sqrt{3}i)}{2}$$

$$\{-2, -2\omega, -2\omega^2\}.$$

(iii) -27

Sol:- Let x be the cube root

$x^3 = -27$

$x^3 + 27 = 0$

$x^3 + 3^3 = 0$

$(x+3)(x^2-3x+9) = 0$

$x+3=0$, $x^2-3x+9=0$

$x=-3$, $a=1$, $b=-3$, $c=9$

, $x = \dfrac{-(-3) \pm \sqrt{(-3)^2-4(1)(9)}}{2(1)}$

, $x = \dfrac{3 \pm \sqrt{9-36}}{2}$

, $x = \dfrac{3 \pm \sqrt{-27}}{2}$

, $x = \dfrac{3 \pm 3\sqrt{-3}}{2}$

$x = 3\left(\dfrac{1 \pm \sqrt{3}\,i}{2}\right)$

$x = -3\left(\dfrac{-1 \pm \sqrt{3}\,i}{2}\right)$

$\{-3, -3\omega, -3\omega^2\}$

(iv) 64

Sol:-

Let x be the cube root

$x^3 = 64$

$x^3 - 64 = 0$

$x^3 - 4^3 = 0$

$(x-4)(x^2+4x+16) = 0$

$x-4=0, \quad x^2+4x+16=0$

$x=4, \quad a=1, \; b=4 \quad c=16$

$x = \frac{-(4) \pm \sqrt{(4)^2-4(1)(16)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{16-64}}{2}$

$x = \frac{-4 \pm \sqrt{-48}}{2}$

$x = \frac{-4 \pm \sqrt{16 \times 3 i^2}}{2}$

$x = \frac{-4 \pm 4\sqrt{3}\, i}{2}$

$x = 4\frac{(-1 \pm \sqrt{3} i)}{2}$

$\{4, \; 4\omega, \; 4\omega^2\}$

(v) -125

Sol:-

Let x be the cube root

$x^3 = -125$

$x^3 + 125 = 0$

$x^3 + 5^3 = 0$

$(x+5)(x^2 - 5x + 25) = 0$

$x + 5 = 0$, $x^2 - 5x + 25 = 0$

$x = -5$, $a = 1$, $b = -5$, $c = 25$

$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(25)}}{2(1)}$

$x = \frac{5 \pm \sqrt{25 - 100}}{2}$

$x = \frac{5 \pm \sqrt{-75}}{2}$

$x = \frac{5 \pm \sqrt{25 \times 3 i^2}}{2}$

$x = \frac{5 \pm 5\sqrt{3}i}{2}$

$x = 5\frac{(1 \pm \sqrt{3}i)}{2}$

$x = -5\frac{(-1 \pm \sqrt{3}i)}{2}$

$\{-5, -5\omega, -5\omega^2\}$

Q2 Find the fourth root of 16, 81, 625. Also show that their sum is zero in each case.

(i) 16

Sol:- Let x be the 4th root

$$x^4 = 16$$
$$x^4 - 16 = 0$$
$$(x^2)^2 - (4)^2 = 0$$
$$(x^2 - 4)(x^2 + 4) = 0$$
$$x^2 - 4 = 0 \quad , \quad x^2 + 4 = 0$$
$$x^2 = 4 \quad , \quad x^2 = -4$$
$$\sqrt{x^2} = \sqrt{4} \quad , \quad \sqrt{x^2} = \sqrt{-4}$$
$$x = \pm 2 \quad , \quad x = \pm 2i$$

Sum $= +2 - 2 + 2i - 2i$

$= 0$

Hence Proved.

(ii) 81

Sol:- Let x be the 4th root

$$x^4 = 81$$
$$x^4 - 81 = 0$$

$x^4 - 3^4 = 0$

$(x^2)^2 - (9)^2 = 0$

$(x^2 - 9)(x^2 + 9) = 0$

$x^2 - 9 = 0 \quad , \quad x^2 + 9 = 0$

$x^2 = 9 \quad , \quad x^2 = -9$

$\sqrt{x^2} = \sqrt{9} \quad , \quad \sqrt{x^2} = \sqrt{-9}$

$x = \pm 3 \quad , \quad x = \pm 3i$

$(3, -3, 3i, -3i)$

Sum $= 3 - 3 + 3i - 3i$

$= 0$

Hence proved.

(ii) 625

Sol:- Let x be the 4th root

$x^4 = 625$

$x^4 - 625 = 0$

$x^4 - 5^4 = 0$

$(x^2)^2 - (25)^2 = 0$

$(x^2 - 25)(x^2 + 25) = 0$

$x^2 - 25 = 0 \quad , \quad x^2 + 25$

$x^2 = 25 \quad , \quad x^2 = -25$

$\sqrt{x^2} = \sqrt{25} \quad , \quad \sqrt{x^2} = \sqrt{-25}$

$x = \pm 5 \quad , \quad x = \pm 5i$

$\{+5, -5, 5i, -5i\}$

Sum $= 5 - 5 + 5i - 5i$

$= 0$

Hence Proved.

Q3:- If $1, \omega, \omega^2$ are the cube roots of unity, show that $1 + \omega^n + \omega^{2n} = 3$ where n is a multiple of 3 respectively.

Sol:- Let ω be a primitive cube root of unity, so:

$\omega^3 = 1$, $1 + \omega + \omega^2 = 0$

Also, cube roots of unity repeat every 3 terms:

$\omega^{3k} = 1$, $\omega^{3k+1} = \omega$, $\omega^{3k+2} = \omega^2$

Let $n = 3k$

Then,

$\omega^n = \omega^{3k} = 1$

$\omega^{2n} = \omega^{6k} = (\omega^3)^{2k} = (1)^{2k} = 1$

Now,

$1 + \omega^n + \omega^{2n} = 1 + 1 + 1$

$= 3$

Hence Proved.

Q4: Evaluate:

(i) $\left(\frac{-1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$

Sol:-

$$= \left(\frac{-1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$$

$$= (\omega)^7 + (\omega^2)^7$$

$$= \omega^7 + \omega^{14}$$

$$= \omega.\omega^6 + \omega^2.\omega^6$$

$$= \omega^6(\omega + \omega^2)$$

$$= (\omega^3)^2(-1) \quad \because \omega + \omega^2 = -1$$

$$= (1)^2(-1)$$

$$= 1(-1)$$

$$= -1$$

(ii) $(-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5$

Sol:-

$$= (-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5$$

$$= \left(2\left(\frac{-1+\sqrt{-3}}{2}\right)\right)^5 + \left(2\left(\frac{-1-\sqrt{-3}}{2}\right)\right)^5$$

$$= 2^5\omega^5 + 2^5(\omega^2)^5$$

$$= 32\,\omega^2.\omega^3 + 32.\omega^1.\omega^9$$

$$= 32\left(\omega^2(1) + \omega(\omega^3)^3\right)$$

$$= 32\left(\omega^2 + \omega(1)^3\right)$$

$= 32(\omega^2+\omega)$

$= 32(-1)$

$= -32$

Q5 Show that

$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots$ to $2n$ factors $= 2^{2n}$.

Sol:-

$LHS = (1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$

$= [(1-\omega+\omega^2)(1-\omega^2+\omega^4)][(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})]$

$= [1-\cancel{\omega^2}+\cancel{\omega^4}-\omega+\omega^3-\omega^5+\cancel{\omega^2}-\cancel{\omega^4}+\omega^6].$
$[1-\cancel{\omega^8}+\cancel{\omega^{16}}-\omega^4+\omega^{12}-\omega^{20}+\cancel{\omega^8}-\cancel{\omega^{16}}+\omega^{24}]$

$= (1-\omega+(1)-\omega^5+\omega^6)(1-\omega^4+\omega^{12}-\omega^{20}+\omega^{24})$

$= (1-\omega+1-\omega^3.\omega^2.+(\omega^3)^2)(1-\omega.\omega^3+(\omega^3)^4-\omega^2.\omega^{18}+(\omega^3)^8)$

$= (2-\omega-\omega^2+1)(1-\omega+1-\omega^2.(\omega^3)^6+1)$

$= (2-(\omega+\omega^2)+1)(2-\omega-\omega^2.+1)$

$= (2-\cancel{1}+\cancel{1})(2-(\omega+\omega^2)+1)$

$= (2)(2-(1)+1)$

$= (2)(2)\ldots\ldots 2n$ factors.

$=$ 4 .... $2n$ factors.

$= (2)^2 \ldots\ldots 2n$ factor

$= 2^{2n}$

$=$ R Hs

Hence proved.

Q6: Prove that

$$\left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8 = -1$$

Sol:-

$$\left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8$$

Let $z = \frac{i+\sqrt{3}}{2}$

and $\bar{z} = \frac{i-\sqrt{3}}{2}$

we have to evaluate $z^8 + \bar{z}^8$

$$z + \bar{z} = \frac{i+\sqrt{3}}{2} + \frac{i-\sqrt{3}}{2}$$

$$= \frac{i}{2} + \frac{\sqrt{3}}{2} + \frac{i}{2} - \frac{\sqrt{3}}{2}$$

$$= \frac{2i}{2} = i$$

So,

$z + \bar{z} = i$

$z \cdot \bar{z} = -1$

As $x^2 - Sx + P = 0$ (Quad eq)

$x^2 - (i)x - 1 = 0$

$x^2 - ix - 1 = 0$

and

$x^2 = ix + 1$

As, $z^1 = z$

$z^2 = iz + 1 \quad \because x^2 = ix + 1$

$z^3 = z(iz+1)$

$= z^2 i + z$

$= (iz+1)i + z \quad \because z^2 = iz+1$

$= i^2 z + i + z$

$= -z + i + z = i$

$z^4 = z \cdot z^3$

$= z \cdot i$

$z^5 = z \cdot (z^4)$

$= z(z \cdot i)$

$= z^2 i \Rightarrow (iz+1)i \Rightarrow zi^2 + i$

$= -z + i$

$z^6 = z \cdot z^5$

$= z(-z+i)$

$= -z^2 + zi$

$= -(iz+1) + zi$

$= -zi - 1 + zi$

$= -1$

$z^7 = z \cdot z^6$

$= z \cdot (-1) = -z$

$z^8 = z^7 \cdot z$

$= -z(z) \Rightarrow -z^2$

$= -(iz+1)$

$= -iz - 1$

and Same for $\bar{z}^8$

$\bar{z}^8 = -i\bar{z} - 1$

$z^8 + \bar{z}^8 = (-iz-1) + (-i\bar{z}-1)$

$= -iz - 1 - i\bar{z} - 1$

$= -i(z+\bar{z}) - 2$

As $z + \bar{z} = i$

$= -i(i) - 2$

$= -i^2 - 2$

$= -(-1) - 2$

$= 1 - 2$

$= -1$

Hence Proved.

Q7:. Evaluate:

$\sum_{K=0}^{5} \omega^{2K}$ where ω is an imaǹ cube root of unity

Sol:.

$\sum_{K=0}^{5} \omega^{2K}$

As $\omega^3 = 1$, and $1+\omega+\omega^2 = 0$

Now calculate,

$= \omega^0 + \omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^{10} \longrightarrow ①$

As

$\omega^0 = 1$, $\omega^2 = \omega^2$, $\omega^4 = \omega$

$\omega^6 = 1$, $\omega^8 = \omega^2$, $\omega^{10} = \omega$

So, Put in ①

$= 1 + \omega^2 + \omega + 1 + \omega^2 + \omega$

$= (1+1) + (\omega + \omega) + (\omega^2 + \omega^2)$

$= 2 + 2\omega + 2\omega^2$

$= 2(1 + \omega + \omega^2)$

$= 2(0)$

$= 0$

Hence Proved.

Q8: If ω is an imag cuberoots of unity Prove that

$$\frac{a + b\omega^2 + c\omega}{a\omega^2 + b\omega + c} = \omega$$

Sol:- $LHS = \dfrac{a + b\omega^2 + c\omega}{a\omega^2 + b\omega + c}$

Multiply both numerator and denominator by ω.

$$= \frac{\omega(a + b\omega^2 + c\omega)}{\omega(a\omega^2 + b\omega + c)}$$

$$= \frac{a\omega + b\omega^3 + c\omega^2}{a\omega^3 + b\omega^2 + c\omega}$$

$$= \frac{a\omega + b + c\omega^2}{a + b\omega^2 + c\omega}$$

$$= \omega \quad = RHS$$

Hence Proved

Q9:- If ω is a cube root of unity, prove that

$$\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}=\omega$$

Sol:-

$$LHS=\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}$$

$\because$ $\omega^{12}=(\omega^3)^4=(1)^4=1$

$\omega^{17}=\omega^2.\omega^{15}\Rightarrow\omega^2.(\omega^3)^5=\omega^2$

$\omega^{19}=\omega^1.\omega^{18}\Rightarrow\omega^1.(\omega^3)^6=\omega$

$\omega^{14}=\omega^2.\omega^{12}\Rightarrow\omega^2.(\omega^3)^4\Rightarrow\omega^2$

$\omega^{22}=\omega.\omega^{21}\Rightarrow\omega.(\omega^3)^7=\omega$

$\omega^{30}=(\omega^3)^{10}=(1)^{10}=1$

Now put all values.

$$=\frac{a(1)+b(\omega^2)+c(\omega)}{a(\omega^2)+b(\omega)+c(1)}$$

$$=\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}$$

Same as previous Question

$=\omega$

$=RHS$

Hence Proved