Unit 1

Complex Numbers

Ex 1.1

Q1: Find the multiplicative inverse of each of the following Complex numbers.

i) $(-4, 7)$

Sol:- By using formula

$$\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(\frac{-4}{(-4)^2+(7)^2}, \frac{-7}{(-4)^2+(7)^2}\right)$$

$$= \left(\frac{-4}{16+49}, \frac{-7}{16+49}\right)$$

$$= \left(\frac{-4}{65}, \frac{-7}{65}\right)$$

(ii) $(\sqrt{2}, -\sqrt{5})$

Sol:- By using formula

$$\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(\frac{\sqrt{2}}{(\sqrt{2})^2+(-\sqrt{5})^2}, \frac{-(-\sqrt{5})}{(\sqrt{2})^2+(-\sqrt{5})^2}\right)$$

$$= \left(\frac{\sqrt{2}}{2+5}, \frac{\sqrt{5}}{2+5} \right)$$

$$= \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7} \right)$$

(iii) $(1, 0)$

Sol:- By using formula

$$\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \right)$$

$$\left(\frac{1}{(1)^2+(0)^2}, \frac{-0}{(1)^2+(0)^2} \right)$$

$$= \left(\frac{1}{1+0}, \frac{0}{1+0} \right)$$

$$= \left(\frac{1}{1}, \frac{0}{1} \right)$$

$$= (1, 0)$$

Q2: Separate into real and Imaginary parts.

(i) $\frac{2-7i}{4+5i}$

Sol:- $\frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$

$$= \frac{(2-7i)(4-5i)}{(4+5i)(4-5i)}$$

$$= \frac{2(4-5i)-7i(4-5i)}{(4)^2-(5i)^2}$$

$$= \frac{8-10i-28i+35i^2}{16-25i^2}$$

$$= \frac{8-38i+35(-1)}{16-25(-1)} \quad \because i^2=-1$$

$$= \frac{8-38i-35}{16+25}$$

$$= \frac{-27-38i}{41}$$

$$= \frac{-27}{41} - \frac{38}{41}i$$

The real part is $\frac{-27}{41}$

and img part is $\frac{-38}{41}$

(ii) $\frac{(-2+3i)^2}{1+i}$

Sol:- $\quad \because (a-b)^2 = a^2-2ab+b^2$

$$= \frac{4-12i+9i^2}{1+i}$$

$$= \frac{4 - 12i + 9(-1)}{1+i} \qquad \because i^2 = -1$$

$$= \frac{4 - 12i - 9}{1+i}$$

$$= \frac{-5 - 12i}{1+i}$$

$$= \frac{-5 - 12i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{(-5 - 12i)(1-i)}{(1+i)(1-i)}$$

$$= \frac{-5(1-i) - 12i(1-i)}{(1)^2 - (i)^2}$$

$$= \frac{-5 + 5i - 12i + 12i^2}{1 - (i^2)}$$

$$= \frac{-5 - 7i + 12(-1)}{1 - (-1)} \qquad \because i^2 = -1$$

$$= \frac{-5 - 7i - 12}{1+1}$$

$$= \frac{-17 - 7i}{2}$$

$$= \frac{-17}{2} - \frac{7i}{2}$$

Real part is $\frac{-17}{2}$ and img part is $\frac{-7}{2}i$.

(iii) $\frac{i}{1+i}$

Sol:-

$: \frac{i}{1+i} \times \frac{1-i}{1-i}$

$\Rightarrow \frac{i(1-i)}{(1+i)(1-i)} \Rightarrow \frac{i-i^2}{1-i^2}$

$\Rightarrow \frac{i-(-1)}{1-(-1)}$ $\because i^2=-1$

$\Rightarrow \frac{i+1}{1+1} = \frac{i+1}{2} \Rightarrow \frac{i}{2} + \frac{1}{2}$

$\Rightarrow \frac{1}{2} + \frac{i}{2}$

Real part is $\frac{1}{2}$ and img part is $\frac{1i}{2}$

(iv) $\frac{(4+3i)^2}{4-3i}$

Sol:-

$$= \frac{(4)^2 + 2(4)(3i) + (3i)^2}{4-3i}$$

$$= \frac{16 + 24i + 9i^2}{4-3i}$$

$$= \frac{16 + 24i + 9(-1)}{4-3i} \quad \because i^2 = -1$$

$$= \frac{16 + 24i - 9}{4-3i}$$

$$= \frac{7 + 24i}{4-3i}$$

$$= \frac{7 + 24i}{4-3i} \times \frac{4+3i}{4+3i}$$

$$= \frac{(7+24i)(4+3i)}{(4-3i)(4+3i)}$$

$$= \frac{7(4+3i) + 24i(4+3i)}{(4)^2 - (3i)^2}$$

$$= \frac{28 + 21i + 96i + 72i^2}{16 - 9i^2}$$

$$= \frac{28 + 117i + 72(-1)}{16 - 9(-1)} \quad \because i^2 = -1$$

$$= \frac{28 + 117i - 72}{16+9}$$

$$= \frac{-44 + 117i}{25}$$

$$= \frac{-44}{25} + \frac{117}{25}i$$

Real part is $\frac{-44}{25}$ and img part is $\frac{117}{25}i$

Q3 Prove that $\bar{z} = z$ iff z is real.

Sol:-

Let $z = a + bi$ then $\bar{z} = a - bi$

Suppose $z = \bar{z}$

$\Rightarrow a + bi = a - bi$

$\Rightarrow a + bi - a + bi = 0$

$\Rightarrow 2bi = 0 \Rightarrow b = 0 \quad \because 2i \neq 0$

$\Rightarrow z = a + i0$

$\Rightarrow z = a$

$\Rightarrow z$ is real.

Now Conversely:

Suppose that z is real

Such that $z = a$

then $\bar{z} = a$

$\Rightarrow z = \bar{z}$

Hence Proved.

Q4: For $z \in C$, Show that

(i) $\frac{z+\bar{z}}{2} = Re(z)$

Sol:-

As $z \in C$, so

$z = a + bi$ and $\bar{z} = a - bi$

$LHS = \frac{z+\bar{z}}{2}$

$= \frac{a+bi+a-bi}{2}$

$= \frac{\cancel{2}a}{\cancel{2}}$

$= a$

$= Re(z)$

$= RHS$

Hence Proved.

(ii) $\frac{z-\bar{z}}{2i} = Im(z)$

Sol:- As $z \in C$ so

$z = a+bi$ and $\bar{z} = a - bi$

$$L.H.S = \frac{z-\bar{z}}{2i}$$

$$= \frac{(a+bi)-(a-bi)}{2i}$$

$$= \frac{a+bi-a+bi}{2i}$$

$$= \frac{2bi}{2i}$$

$$= b$$

$$= Img(z)$$

$$= RHS$$

Hence Proved.

(iii) $|z|^2 = z.\bar{z}$

Sol:-

As $z \in C$, so

$z = a+bi$ and $\bar{z} = a-bi$

$$LHS = |z|^2 = |a+bi|^2$$

$$= \left(\sqrt{(a)^2+(b)^2}\right)^2 = \left(\sqrt{a^2+b^2}\right)^2$$

$$= a^2+b^2$$

$$RHS = z.\bar{z} = (a+bi)(a-bi)$$

$$= (a)^2-(bi)^2 = a^2-b^2i^2$$

$$= a^2-b^2(-1) = a^2+b^2$$

$$LHS = R.H.S$$

Hence Proved.

Q5 If $z_1 = 2+i$, $z_2 = 3-2i$, $z_3 = 1+3i$ then express $\frac{\bar{z}_1 \bar{z}_3}{z_2}$ in the form of $a+bi$.

Sol:-

$z_1 = 2+i$, $z_2 = 3-2i$, $z_3 = 1+3i$

and

$\bar{z}_1 = 2-i$, $\bar{z}_3 = 1-3i$

$$\frac{\bar{z}_1 \bar{z}_3}{z_2} = \frac{(2-i)(1-3i)}{3-2i}$$

$$= \frac{2(1-3i) - i(1-3i)}{3-2i}$$

$$= \frac{2-6i-i+3i^2}{3-2i}$$

$$\frac{2-7i-3}{3-2i} = \frac{-1-7i}{3-2i}$$

$$= \frac{-1-7i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{(-1-7i)(3+2i)}{(3-2i)(3+2i)}$$

$$= \frac{-1(3+2i) - 7i(3+2i)}{(3)^2-(2i)^2}$$

$$= \frac{-3 - 2i - 21i - 14i^2}{9 - 4i^2}$$

$$= \frac{-3 - 23i + 14}{9 + 4} \qquad \because i^2 = -1$$

$$= \frac{11 - 23i}{13}$$

$$= \frac{11}{13} - \frac{23}{13}i$$

Q6 If $z_1 = 2 + 7i$ and $z_2 = -5 + 3i$, then evaluate the following.

(i) $|2z_1 - 4z_2|$

Sol:-

$$|2z_1 - 4z_2| = |2(2+7i) - 4(-5+3i)|$$

$$= |(4 + 14i) - (-20 + 12i)|$$

$$= |4 + 14i + 20 - 12i|$$

$$= |24 + 2i|$$

$$= \sqrt{(24)^2 + (2)^2}$$

$$= \sqrt{576 + 4}$$

$$= \sqrt{580}$$

$$= \sqrt{4 \times 145}$$

$$= 2\sqrt{145}$$

(ii) $|3z_1 + 2\bar{z}_1|$

Sol:-

As $z_1 = 2+7i$ and $\bar{z}_1 = 2-7i$

$= |3(2+7i) + 2(2-7i)|$

$= |6 + 21i + 4 - 14i|$

$= |10 + 7i|$

$= \sqrt{(10)^2 + (7)^2}$

$= \sqrt{100 + 49}$

$= \sqrt{149}$

(iii) $|-7z_2 + 2\bar{z}_2|$

Sol:-

As $z_2 = -5+3i$ and $\bar{z}_2 = -5-3i$

$= |-7(-5+3i) + 2(-5-3i)|$

$= |35 - 21i - 10 - 6i|$

$= |25 - 27i|$

$= \sqrt{(25)^2 + (-27)^2}$

$= \sqrt{625 + 729}$

$= \sqrt{1354}$

(iv) $|(z_1+z_2)^3|$

Sol:- We are given two complex no

As $z_1 = 2+7i$ and $z_2 = -5+3i$

$= |z_1 + z_2|^3$

$$|z_1+z_2|^3 = |(2+7i)+(-5+3i)|^3$$
$$= |2+7i-5+3i|^3$$
$$= |-3+10i|^3$$
$$|z_1+z_2|^3 = |-3+10i|^3$$
$$= \left(\sqrt{(-3)^2+(10)^2}\right)^3$$
$$= \left(\sqrt{9+100}\right)^3$$
$$= \left(\sqrt{109}\right)^3$$
$$= \left(\sqrt{109}\right)^2 . \left(\sqrt{109}\right)$$
$$= 109.\sqrt{109}$$

Q7 Show that

$$i^{n+1}+i^{n+2}+i^{n+3}+i^{n+4}=0 \text{ for all}$$

$n \in N$.

Sol:-

$$LHS = i^{n+1}+i^{n+2}+i^{n+3}+i^{n+4}$$
$$= i^n.i + i^n.i^2 + i^n.i^3 + i^n.i^4$$
$$= i^n(i+i^2+i^3+i^4)$$
$$= i^n(i+(-1)+i.i^2+(i^2)^2)$$
$$= i^n(i-1+i(-1)+(-1)^2)$$
$$= i^n(i-1-i+1)$$
$$= i^n(0)$$
$$= 0$$
$$= RHS$$

Hence Proved.

Q8 Find the least positive ~~value~~ value of n, if

$$\left(\frac{1+i}{1-i}\right)^{2n} = 1$$

Sol:-
$$\left(\frac{1+i}{1-i}\right)^{2n} = 1$$

$$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^{2n} = 1$$

$$\left(\frac{(1+i)^2}{(1)^2-(i)^2}\right)^{2n} = 1$$

$$\left(\frac{1+i^2+2i}{1-(-1)}\right)^{2n} = 1$$

$$\left(\frac{1-1+2i}{1+1}\right)^{2n} = 1$$

$$\left(\frac{2i}{2}\right)^{2n} = 1$$

$$(i)^{2n} = 1$$

$$(i)^{2n} = (i)^0$$

So, $\Rightarrow 2n = 0 \pmod 4$

$\Rightarrow n = 0 \pmod 2$

So least positive value of n will be 2

$$\boxed{n=2}$$

Q9:- Show that, the value of i^n for $n \in N$ and $n > 4$ is i^r, wher r is the remainder when n is divided by 4.

Sol:-

Divide n by 4

$n = 4K + r$, where $r = 0, 1, 2, 3$

Here

K is the quotient

r is the remainder

Put into i^n

$i^n = i^{4K+r}$

Use exponent rules:

$i^{4K+r} = i^{4K} \cdot i^r$

But we know

$i^4 = 1 \Rightarrow i^{4K} \Rightarrow (1)^K = 1$

So,

$i^n = 1 \cdot i^r = i^r$

$i^n = i^r$

Hence Proved.